Semi-Lagrangian schemes for second order Mean Field Game systems

Francisco J. Silva

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F. J. Silva
XLIM-Université de Limoges

based on a collaboration with
E. Carlini
Università Sapienza di Roma

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Outline

Mean Field Game Problem

Model introduced by J.M.Lasry & P.L.Lions 2006

$$\begin{cases} -\partial_t v(x,t) - \sigma^2 \Delta v(x,t) + H(x, Dv(x,t)) = F(x, m(t)), & \mathbb{R}^d \times (0,T), \\ \partial_t m(x,t) - \sigma^2 \Delta m(x,t) - \operatorname{div}\big(\partial_p H(x, Dv(x,t)) m(x,t)\big) = 0, & \mathbb{R}^d \times (0,T), \\ v(x,T) = G(x, m(T)) \quad \text{for } x \in \mathbb{R}^d, \quad m(0) = m_0 \in \mathcal{P}_1, \end{cases}$$

- $\sigma \geq 0 \in \mathbb{R}$ and $H(x, \cdot)$ is convex.
- in the 1st line we have a Hamilton Jacobi (HJ) equation backward in time. It can be represented as the value function of an associated optimal control problem.

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- $\sigma \geq 0 \in \mathbb{R}$ and $H(x, \cdot)$ is convex.
- in the 1st line we have a Hamilton Jacobi (HJ) equation backward in time. It can be represented as the value function of an associated optimal control problem.
- in the 2nd line we have a Fokker Plank equation forward in time
- m_0 and then $m(x,t)$ represents the density of a probability measure, evolving with the velocity field $\partial_p H(x, Dv(x,t))$.

Mean Field Problem

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$$
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\partial_t m(x,t) - \sigma^2 \Delta m(x,t) - \operatorname{div}\big(\partial_p H(x, Dv(x,t)) m(x,t)\big) = 0, & \mathbb{R}^d \times (0,T), \\
v(x,T) = G(x, m(T)) \quad \text{for } x \in \mathbb{R}^d, \quad m(0) = m_0 \in \mathcal{P}_1,
\end{cases}
$$

Our aim: to provide a fully discrete approximation such that: when the discretization parameter tends to 0, we have the convergence to (v, m).

Some references of Numerical Approximation of MFG

- Second order problem ($\sigma \neq 0$)
  - Y.Achdou, I.Capuzzo-Dolcetta ('10),
    Y.Achdou, F.Camilli, I.Capuzzo-Dolcetta ('12)
    (Semi-implicite Finite Difference scheme, Newton Iteration)
  - A.Lachapelle, J.Salomon, G. Turinici ('10) (monotonic scheme)
  - O.Gueant (finite difference for a two linear parabolic equations obtained by a change of variable)

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  - A.Lachapelle, J.Salomon, G. Turinici ('10) (monotonic scheme)
  - O.Gueant (finite difference for a two linear parabolic equations obtained by a change of variable)
- First order problem ($\sigma = 0$)
  - Y.Achdou, F.Camilli, L.Corrias ('12) (Semi-Lagrangian scheme, forward-forward system, coupling only on the second one)
  - F.Camilli and S-('12) (semi-discrete Semi-Lagrangian scheme)
  - E. Carlini and S-('13) (fully-discrete Semi-Lagrangian scheme)

A fully discrete SL scheme for a possibly degenerated second order MFG system

We consider the system system

$$
\begin{cases}
-\partial_t v - \mathrm{tr}(\sigma(t)\sigma^\top(t)D^2 v) + \frac{1}{2}|Dv|^2 = F(x, m(t)), \;\; \text{in } Q, \\
\partial_t m - \mathrm{tr}(\sigma(t)\sigma^\top(t)D^2 m) - \mathrm{div}\big(Dvm\big) = 0, \quad \text{in } Q, \\
v(x,T) = G(x, m(T)) \quad \text{for } x \in \mathbb{R}^d, \;\; m(0) = m_0.
\end{cases}
\tag{1}
$$

where $\sigma : [0,T] \to \mathbb{R}^{d\times r}$ is bounded and continuous.

A fully discrete SL scheme for the second order MFG system

We present the scheme for the simplest equation with only one Brownian motion

$$\begin{cases} -\partial_t v - \frac{1}{2}\sigma^2(t)\Delta v + \frac{1}{2}|Dv|^2 = F(x, m(t)), \ \text{ in } Q, \\ \partial_t m - \frac{1}{2}\sigma^2(t)\Delta m - \text{div}\big(Dvm\big) = 0, \quad \text{in } Q, \\ v(x,T) = G(x, m(T)) \quad \text{for } x \in \mathbb{R}, \ \ m(0) = m_0. \end{cases} \tag{2}$$

The discretization of the HJB equation

The solution of the first equation can be represented as the value function of the a stochastic optimal control problem. More precisely, given $\mu \in C([0,T];\mathcal{P}_1)$ let $v[\mu]$ be the unique viscosity solution of

$$\begin{aligned} -\partial_t v - \tfrac{1}{2}\sigma^2(t)\Delta v + \tfrac{1}{2}|Dv|^2 &= F(x,\mu(t)), \quad \text{in } \mathbb{R}^d \times [0,T], \\ v(x,T) &= G(x,\mu(T)) \quad \text{for } x \in \mathbb{R}^d. \end{aligned}$$

Then $v[\mu]$ can be represented as the value function of the following stochastic optimal control problem

$$v[\mu](x,t) =$$

$$\inf_{\alpha \in L^{2,2}_{\mathbb{F}}} \mathbb{E}\left(\int_t^T \left[\tfrac{1}{2}|\alpha(s)|^2 + F(X^{x,t}[\alpha](s),\mu(s)\right] \mathrm{d}s + G(X^{x,t}[\alpha](T),\mu(T)) \right),$$

where

$$X^{x,t}[\alpha](s) = x - \int_t^s \alpha(s)\mathrm{d}s + \int_t^s \sigma(s)\mathrm{d}W(s), \quad \forall\, s \in [t,T].$$

The value function is Lipschitz w.r.t. x and $\frac{1}{2}$-Holder continuous w.r.t. t. Therefore, it satisfies the following DPP:

$$v[\mu](x,t) = \inf_{\alpha\in L^{2,2}_{\mathbb{F}}} \mathbb{E}\left(\int_t^{t+h}\left[\tfrac{1}{2}|\alpha(s)|^2 + F(X^{x,t}[\alpha](s),\mu(s)\right]\mathrm{d}s + v(X^{x,t}[\alpha](t+h),t+h)\right)$$

for all $h\in[0,T-t]$. This induces the following natural discretization (see Camilli-Falcone '95).

$$\begin{cases} v_{i,k} = \hat{S}_{\rho,h}[\mu](v_{\cdot,k+1},i,k) \quad \forall\, k=0,\dots,N-1,\\ v_{i,N} = G(x_i,\mu(t_N)), \end{cases}$$

where $\hat{S}_{\rho,h}[\mu] : B(\mathcal{G}_\rho) \times \mathbb{Z} \times \{0, \dots, N-1\} \to \mathbb{R}$ is defined as

$$\hat{S}_{\rho,h}[\mu](f,i,k) := \inf_{\alpha\in\mathbb{R}} \left(\frac{1}{2} I[f](x_i - h\alpha + \sqrt{h}\sigma(t_k) + \frac{1}{2} I[f](x_i - h\alpha - \sqrt{h}\sigma(t_k)) + \tfrac{1}{2} h|\alpha|^2 \right) + hF(x_i, \mu(t_k)).$$

We have the following properties of the scheme:

Proposition

The scheme is well defined, monotone and consistent.

We extend $v(\cdot,\cdot)$ to $\mathbb{R}$ interpolating in space and constant if the interval $[t_k, t_k + h]$ and we denote the new function as $v_{\rho,h}(\cdot,\cdot)$.

We have that

Lemma

For every $t \in [0,T]$, the following assertions hold true:

(i) [Lipschitz property] *The function $v_{\rho,h}[\mu](\cdot,t)$ is Lipschitz with constant independent of (ρ,h,μ,t).*

(ii) [Discrete semiconcavity] *There exists $c>0$ independent of (ρ,h,μ,t) such that*

$$v_{\rho,h}[\mu](x_i+x_j,t)-2v_{\rho,h}[\mu](x_i,t)+v_{\rho,h}[\mu](x_i-x_j,t)\leq c|x_j|^2$$

Given $\varepsilon > 0$ and $\phi \in C_0^\infty(\mathbb{R}^d)$, with $\phi \geq 0$ and $\int_{\mathbb{R}^d} \phi(x)\mathrm{d}x = 1$, we set $\phi_\varepsilon(x) := \frac{1}{\varepsilon^d}\phi(x/\varepsilon)$ and

$$v^\varepsilon_{\rho,h}[\mu](\cdot,t) := \phi_\varepsilon * v_{\rho,h}[\mu](\cdot,t) \quad \forall\, t \in [0,T].$$

The following lemma is crucial (see e.g. Achdou-Camilli-Corrias'13)

Lemma

For every $t \in [0,T]$ we have that:

(i) *The function $v^\varepsilon_{\rho,h}[\mu](\cdot,t)$ is Lipschitz, independently of (ρ,h,μ,t).*

(ii) *There exists $c > 0$ independent of $(\rho,h,\varepsilon,\mu,t)$, such that*

$$\langle Dv^\varepsilon_{\rho,h}[\mu](y,t) - Dv^\varepsilon_{\rho,h}[\mu](x,t), y-x\rangle \leq c\left(|x-y|^2 + \left(\frac{\rho}{\varepsilon}\right)^2\right).$$

We have the following stability result:

Theorem

Let $(\rho_n, h_n, \varepsilon_n) \to 0$ *be such that* $\frac{\rho_n^2}{h_n} \to 0$ *and* $\rho_n = o(\varepsilon_n)$. *Then, for every sequence* $\mu_n \in C([0,T];\mathcal{P}_1)$ *such that* $\mu_n \to \mu$ *in* $C([0,T];\mathcal{P}_1)$, *we have that* $v_{\rho_n,h_n}^{\varepsilon_n}[\mu_n] \to v[\mu]$ *uniformly over compact sets and* $Dv_{\rho_n,h_n}^{\varepsilon_n}[\mu_n](x,t) \to Dv[\mu](x,t)$ *at every* (x,t) *such that* $Dv[\mu](x,t)$ *exists.*

- The convergence of the value functions can be obtained by slightly modifying the scheme or the proof in the Barles-Souganidis results '91.
- The a.s. convergence of the derivatives is a consequence of the semiconcavity estimate assured by the previous Lemma.

A fully-discrete scheme for the Fokker-Planck equation

Given a regular velocity field $b : \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}$, consider the Fokker Planck equation

$$\begin{aligned} \partial_t m - \tfrac{1}{2}\sigma^2(t)\Delta m + \mathrm{div}\big(bm\big) &= 0, \quad \text{in } Q, \\ m(0) &= m_0. \end{aligned}$$

It is well known that the solution of this equation can be represented as

$$m(t) = \mathrm{Law}(X(t)) \;\text{ given that } \mathrm{Law}(X(0)) = m_0,$$

where

$$X(t) = X(0) + \int_0^t b(X(s), s)\mathrm{d}s + \int_0^t \sigma(s)\mathrm{d}W(s).$$

Equivalently,

$$m(t)(A) = \mathbb{E}_{\mathbb{P}}\left(\Phi(\cdot,t,\omega)\sharp m_0(A)\right), \quad \forall\, A \in \mathcal{B}(\mathbb{R}^d),$$

where $\mathbb{P}$ is the Wiener measure in $C([0,T];\mathbb{R}^d)$ and $\Phi(x,\cdot,\omega)$ is the solution of the following SDE

$$\begin{array}{rcl} \mathrm{d}\Phi(x,t) & = & b(\Phi(x,t),t)\mathrm{d}t + \sigma(t)\mathrm{d}W(t), \\ \Phi(x,0) & = & x. \end{array}$$

In our case, $b = -Dv$ which is not regular. However, the discretization we propose is naturally inspired by the above interpretation.

Given μ we let

$$\Phi^{\varepsilon,+}_{i,k}[\mu] := x_i - hDv^{\varepsilon}_{\rho,h}[\mu](x_i,t_k) + \sigma(t_k)\sqrt{h},$$
$$\Phi^{\varepsilon,-}_{i,k}[\mu] := x_i - hDv^{\varepsilon}_{\rho,h}[\mu](x_i,t_k) - \sigma(t_k)\sqrt{h}.$$

and we consider the following scheme for the Fokker Planck equation with $b(x,t) = Dv^{\varepsilon}_{\rho,h}(x,t)$

$$\begin{cases} m^{\varepsilon}_{i,k+1}[\mu] := & \frac{1}{2}\sum_{j\in\mathbb{Z}} \beta_i\left(\Phi^{\varepsilon,+}_{j,k}[\mu]\right) m^{\varepsilon}_{j,k}[\mu] \\ & +\frac{1}{2}\sum_{j\in\mathbb{Z}} \beta_i\left(\Phi^{\varepsilon,-}_{j,k}[\mu]\right) m^{\varepsilon}_{j,k}[\mu], \\ m^{\varepsilon}_{i,0}[\mu] := & \int_{E_i} m_0(x)\mathrm{d}x, \end{cases}$$

Remark (Probabilistic interpretation)

Let us define

$$\begin{array}{rcl} p_{j,i}^{(k)} & := & \frac{1}{2}\left[\beta_i\left(\Phi_{j,k}^{\varepsilon,+}[\mu]\right)+\beta_i\left(\Phi_{j,k}^{\varepsilon,-}[\mu]\right)\right], \quad \forall\, k=0,\dots,N-1, \\ p_i^{(0)} & = & m_{i,0}[\mu]. \end{array} \tag{3}$$

Classical results in probability theory implies that the family $\{p_{j,i}^{(k)}\ ;\ j,\ i\in\mathbb{Z}^d,\ k=0,\dots,N-1\}$ *together with* $\{p_i^{(0)}\ ;\ i\in\mathbb{Z}^d\}$ *allow to define a probability space* $(\Omega,\mathcal{F},\mathbb{P})$ *and a discrete Markov chain* $(X_k)_{0\le k\le N}$ *taking values in* $\mathbb{Z}^d$*, such that its initial distribution is given by* $(p_i^{(0)})_{i\in\mathbb{Z}^d}$*, the transition probabilities are given by* (3) *and the law at time* k *is given by* $m_{i,k}^{\varepsilon}$*.*

The time dependent measure defined above is extended uniformly in each cell in the space and by linear interpolation in time. We denote it by $m^{\varepsilon}_{\rho,h}(\cdot,\cdot)$. We have the following results:

Proposition

Suppose that $\rho = O(h)$. Then, there exists a constant $c > 0$ (independent of $(\rho, h, \varepsilon, \mu)$) such that for all $0 \leq s \leq t \leq T$, we have that

$$\mathbf{d}_1 \left(m^{\varepsilon}_{\rho,h}[\mu](t), m^{\varepsilon}_{\rho,h}[\mu](s) \right) \leq c\sqrt{t-s}. \tag{4}$$

Proposition

If $\rho = O(\sqrt{h})$, then there exists $c > 0$ (independent of $(\rho, h, \varepsilon, \mu)$) such that

$$\int_{\mathbb{R}^d} |x|^2 \mathrm{d}m^{\varepsilon}_{\rho,h}[\mu](t) \leq c \quad \forall\, t \in [0,T]. \tag{5}$$

We recall

Lemma

Let $r > p \geq 1$ and $\mathcal{K} \subseteq \mathcal{P}_p(\mathbb{R}^d)$ be such that

$$\sup_{\mu \in \mathcal{K}} \int_{\mathbb{R}^d} |x|^r \mathrm{d}\mu(x) < \infty.$$

Then the set $\mathcal{K}$ is tight. If moreover $r > p$, then $\mathcal{K}$ is relatively compact for the d_p distance.

Using this fact, the above propositions and Ascoli's theorem we see that if the parameters tend to zero conveniently, we will have limit points.

In order to pass to the limit we will need L^∞ bounds.

Proposition

If $d = 1$ and consider a sequence of positive numbers $(\rho_n, h_n, \varepsilon_n) \to 0$, satisfying that $\rho_n = O(\varepsilon_n^2)$. Then, there exists a constant $c > 0$, independent of (n, μ) such that

$$\|m^{\varepsilon_n}_{\rho_n,h_n}[\mu](\cdot,t)\|_\infty \leq c. \tag{6}$$

- The key in the proof of the above results is the uniform semiconcavity of $v^{\varepsilon}_{\rho,h}[\mu](\cdot,\cdot)$ if $\rho_n = O(\varepsilon_n^2)$.
- It is not clear at all that the above result can be proved for $d > 1$.

The fully discretization of the MFG reads

$$\text{Find } \mu \in C([0,T];\mathcal{P}_1) \text{ such that } m^{\varepsilon}_{\rho,h}[\mu] = \mu. \qquad (MFG)^{\varepsilon}_{\rho,h}$$

Theorem

Problem $(MFG)^{\varepsilon}_{\rho,h}$ has at least one solution.

Theorem

Suppose that $d=1$ and that $\rho_n = o\left(h_n\right)$, $h_n = o(\varepsilon_n^2)$, as $\varepsilon_n \to 0$. Let $\{m^n\}_{n\in\mathbb{N}}$ be a sequence of solutions of $(MFG)_{h,\rho}$ for the corresponding parameters $\rho_n, h_n, \varepsilon_n$. Then any limit point in $C([0,T];\mathcal{P}_1)$ of discrete solutions $m^{\varepsilon_n}_{\rho_n,h_n}$ (there exists at least one) solves (MFG). In particular, if the continuous problem has a unique solution (v,m), then $m^{\varepsilon_n}_{\rho_n,h_n} \to m$ in $C([0,T];\mathcal{P}_1)$ and in $L^{\infty}\left(\mathbb{R}^d \times [0,T]\right)$-weak-$*$.

Numerical test

$\Omega = [0,1]$ and a final time $T = 2$.
Final cost: $G = 0$
Running cost:
$\frac{1}{2}\alpha^2 + F(x,m) = \frac{1}{2}\alpha^2 + 5(x - (1 - \sin(2\pi t))/2)^2 - 10\int_\Omega (y-x)^2 \mathrm{d}m(y)$
Initial mass distribution:

$$m_0(x) = \frac{\nu(x)}{\int_\Omega \nu(x)dx} \quad \text{with} \quad \nu(x) = e^{-(x-0.5)^2/(0.1)^2}$$

and as regularizing kernel $\rho_\varepsilon(x) = \frac{1}{\sqrt{2\pi}} e^{-(x-0.5)^2/2\varepsilon^2}$, with $\varepsilon = 0.3$. We take as space discretization step $\rho = 1.6 \cdot 10^{-2}$ and as time step $h = 0.02$.

Mass distribution First order MFG with $\sigma = 0$

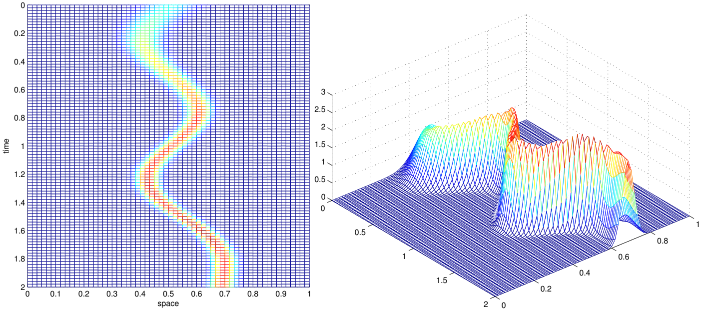

Figure: **Density distribution** $m^{\varepsilon}_{i,k}$

Mass distribution Second order MFG with $\sigma \neq 0$

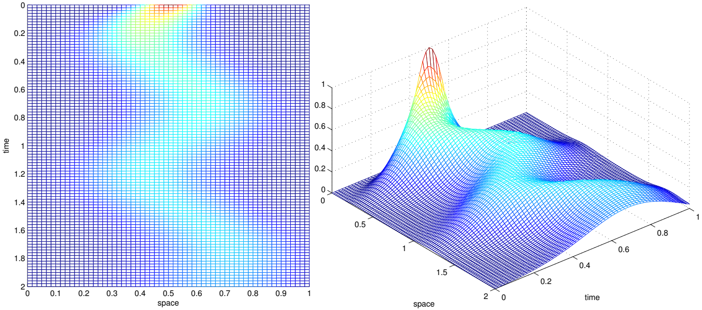

Figure: **Density distribution** $m^{\varepsilon}_{i,k}$

Numerical test second order MFG with $\sigma(t)$ degenerating

$\Omega = [0,1]$ and a final time $T = 2$.
Final cost: $G = 0$
Initial mass distribution:
$m_0(x) = \frac{\nu(x)}{\int_\Omega \nu(x)dx}$ with $\nu(x) = e^{-(x-0.5)^2/(0.1)^2}$
Running cost: $\frac{1}{2}\alpha^2 + F(x,m) = \frac{1}{2}\alpha^2 + f(x,t) + \rho_\delta * [\rho_\delta * m(t)](x)$
with $\delta = 0.25$
and

$$f(x,t) = 5(x - (1 - \sin(2\pi t))/2)^2.$$

We take as space discretization step $\rho = 3.12 \cdot 10^{-3}$ and as time step $h = \rho$ and $\varepsilon = 0.01$.
Degenerate diffusion: $\sigma(t) = (\max(0, 0.2 - |t-1|))^{0.5}$, note that $\sigma(t) = 0$ for all $t \in [0, 0.8] \cup [1.2, 2]$.

Mass distribution first order MFG ($\sigma(t)$)

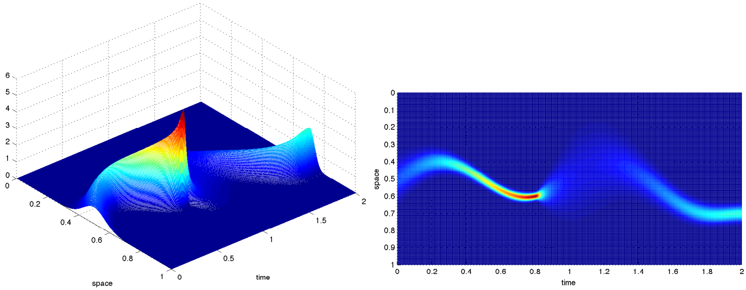

Figure: **Mass distribution** $m_{i,k}^{\varepsilon}$

Numerical test

$\Omega = [-0.1, 1.1]^2$ and a final time $T = 2$.
Final cost: $G = 0$
Running cost: $\frac{1}{2}\alpha^2 + F(x,m) =$
$\frac{1}{2}\alpha^2 + (x-0.75)^2 + (y-0.75)^2 + \beta \int_\Omega \mathcal{I}_{B_r}(y-x)\mathrm{d}m(y)$
Initial mass distribution:

$$m_0(x) = \frac{\nu(x)}{\int_\Omega \nu(x)dx} \quad \text{with}$$

$$\begin{aligned}\nu(x) \quad &= \quad \max(0.05 - (x-0.25)^2 - (y-0.25)^2, 0) \\ &\quad + \max(0.05 - (x-0.75)^2 - (y-0.2)^2, 0)\end{aligned}$$

We take as space discretization step $\rho = 5.5 \cdot 10^{-2}$ as time step $h = 1.3 \cdot 10^{-2}$ and we set $r = \sqrt{\rho}$ (NO parabolic CFL condition is required)

Mass distribution second order MFG with $\sigma = 0.05$ and $\beta = 10$.

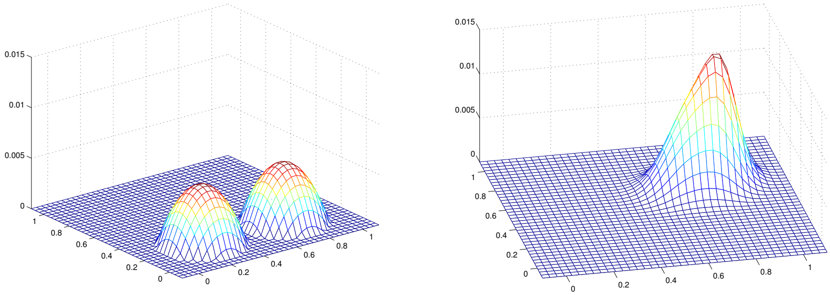

Figure: **Density distribution**

Mass distribution second order MFG with $\sigma = 0.01$ and $\beta = 10$.

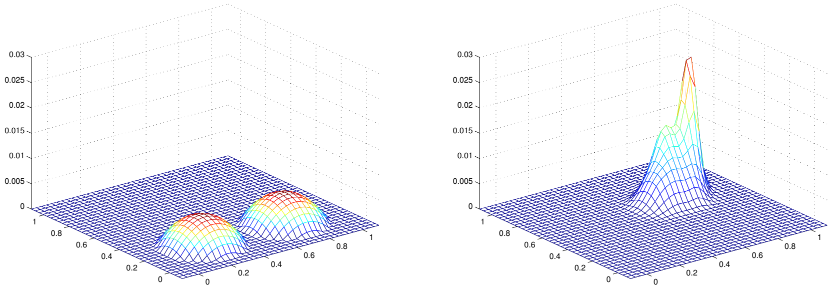

Figure: **Density distribution**

Mass distribution second order MFG with $\sigma = 0.01$ and $\beta = 10$.

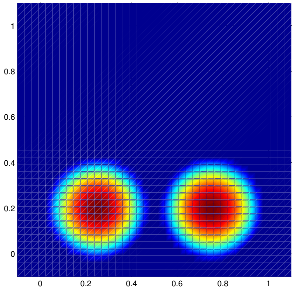

Figure: **Density distribution**

Mass distribution case no game ($\beta = 0$), with $\sigma = 0.01$

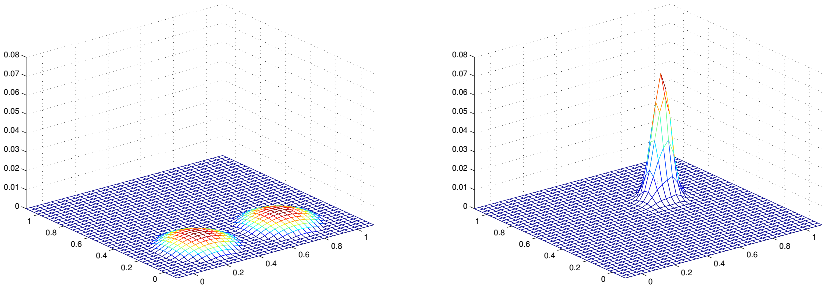

Figure: **Density distribution**

Mass distribution case no game ($\beta = 0$), with $\sigma = 0.01$

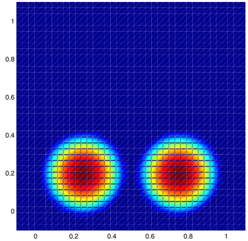

Figure: **Density distribution (right)**

Work in progress and Future works

If $\mathrm{Tr}(\sigma(t)\sigma^{\top}(t))$ is a uniformly elliptic operator, then the convergence result holds true on arbitrary dimensions (no L^{∞} bounds are needed in the proof) and the condition between h and ε is improved to $h = o(\varepsilon)$.

Work in progress

- The non-degenerate case (classical solutions) in arbitrary dimension with σ that depends on x.

Future work

- Extend the Semi-Lagrangian scheme to more general Fokker Plank Equation
- Apply an efficient numerical method to solve the non linear system.

Some References

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